INVARIANCE AS A CRITERION OF REALITY*

F. BONSACK

Summary

A concept of reality is sketched which claims to be metaphysically neutral. It is based on the criterion of invariance, invariance with respect to the observer, the reference frame, the state of the subject, time, mode of perception, kind of experience or language. This criterion is here used negatively : all that is invariant is not neccssarily real, but in the explaining schema we make of the world, that which changes depending on the point of view, the mode of perception or the knowledge of the subject cannot be considered real. The search for invariants allows efficient storage of information and prevision of events which have never been experienced. The concept of 'invariant limit' is introduced and the need of a suitable topology is stressed in order to speak of convergence of successive theories toward reality.

Résumé

Est esquissée une conception de la réalité qui prétend être métaphysiquement neutre. Elle repose sur le critère d'invariance, invariance par rapport à l'observateur, au système de référence, à l'état du sujet, au temps, aux circonstances, au mode de perception, au type d'expérience, au langage. Ce critère est pris ici négativement : tout ce qui est invariant n'est pas forcément réel, mais ce qui, dans les schémas explicatifs que nous nous faisons du monde, varie selon le point de vue, le mode d'appréhension, l'information du sujet ne peut pas être considéré comme réel. La recherche des invariants permet un stockage économique des informations et une prévision de phénomènes qui n'ont jamais été expérimentés. On introduit la notion de limite invariante et on insiste sur la nécessité de définir une bonne topologie si l'on veut parler de convergence des théories successives vers la réalité.

Zusammenfassung

Eine Auffassung der Wirklichkeit wird entworfen, die Anspruch auf metaphysische Neutralität erhebt. Sie stützt sich auf das Kriterium der Invarianz gegenüber dem Beobachter, dem Bezugssystem, dem Zustand des Subjekts, der Zeit, der Umstände, der Perzeptionsart, dem experimentellen Verfahren, der Sprache. Dieses Kriterium ist hier negativ gemeint : alles Invariante ist nicht notwendigerweise wirklich, aber der Teil unserer Vorstellung von der Welt, der vom Standpunkt, vom Erfassungsmittel, vom Wissen des Subjektes abhängt, kann nicht als real betrachtet werden. Die Suche nach Invarianten erlaubt eine ökonomische Speicherung der Information und eine Voraussage von noch nicht beobachteten Phänomenen. Der Begriff «invarianter Grenzwert» wird eingeführt und Gewicht wird darauf gelegt, dass man nicht von Konvergenz der aufeinanderfolgenden Theorien auf die Wirklichkeit hin sprechen darf, solange man keine angemessene Topologie definiert hat.

1. *Definition of realism*

The definitions of realism which one finds in philosophical encyclopedias and dictionaries have certain common features.

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a) Mention is made of an “extemal world” of “reality” (Wirklichkeit), of “being”. This extemal world is considered to be “objective”, “real” or “material”.

b) All definitions insist on the fact that this external world is “independent” of knowledge, sense experience, perception, thought, consciousness and of subject. Here for example is one of the best definitions :

“Standpunkt, wonach es eine vom erkennenden Subjekt (vom Denken, Erkennen, Bewusstsein) unabhängige, selbstseiende, in diesem Sinne absolut seiende (nicht bloss ideellen) Aussenwelt gibt.”[1]

Point *a)* raises considerable difficulties : nothing has been explained as long as the meaning of the words “extemal”, “objective”, “real” or “material” has not been defined.

Point *b)* also raises problems, but these seem easier to approach and to delimit : the meaning of the world ‘independent’ is not as vague as that of ‘real’, or ‘objective’. We shall therefore begin with the study of point *b)*, perhaps occasionally coming back to point *a)*.

2. *How does one know ?*

We will admit, as Gonseth does, that we “know” with the help of schema — the now classical example being that of the ball in the forest.[2]

A schema is a schema of something, either of experience or of certain features of a schema, itself resting more or less directly on experience. The word ‘experience’ is here used in a very broad sense : it includes not only the sensory data, more or less worked out by the nervous centers, but also the subject’s activity to get new information (manipulation, inspection, simple or sophisticated experimentation).

In knowledge certain elements of experience are related to certain elements of the schema. The schema allows the deduction, either of other elements that are not immediately perceived, or the temporal evolution of the schematic situation; this evolution being modified or not by interferences.

These deduced elements can be confronted with active or passive experience : when these schematic elements again correspond with experiment, the schema is said to be suitable; if not, the schema must be revised or at least restricted to a smaller domain than previously thought.

A mathematical theory such as euclidean geometry or a physical theory such as classical mechanics are examples of schema.

This view of knowledge needs some comment. One of the most difficult

[1] E. EISLER : *Wörterbuch der philosophischen Begriffe*, Mittler & Sohn, Berlin (1929).

[2] F. GONSETH : *Les Mathématiques et la Réalité*, Blanchard, Paris (1936, reprinted 1974), p. 226-231.

problems is to relate the elements of the schema with those of experience. As an example is in the schema something called a ‘straight line’; one must specify to which element of experience this element refers, for instance, how a straight line can be drawn, how one can recognize that a point lies on a line, what deviation is accepted and so on. The same schema can be applied to different models, provided the correspondance rules are changed. One of the most striking examples of this ambivalence of schema is that of projective plane geometry — where, in any theorem, the terms ‘straight line’ and ‘point’ can be interchanged without affecting the truth value of the theorem. Thus the same element of the schema can be related to either the experimental element ‘straight line’ or the experimental element ‘point’.

The fact that the correspondence is often only approximate should be emphasized. Schema can take more primitive forms, much less precise than the form ‘scientific theory’. Space, in which we locate perceived objects, and the objects themselves are already schema.

3. *Objections to realism*

One can indeed wonder how the suitability knowledge can be tested, since the external world — if such exists — shows itself only through knowledge. To test the adequacy of our knowledge of the world we ought to compare this knowledge with the original. But we don’t have access to the original : we only dispose of knowledge of the original. The only thing we can then do is to compare one piece of know1edge with another. We are condemned never to know how the world really is.

Ultimately one can doubt the opportuneness of postulating such an “external world”. Cannot one be satisfied with knowledge ? Is the assumption of objects behind the ideas we have of them not a useless and risky duplication ? A duplication that was inherited from naive realism, from which any philosopher should liberate himself ? “Wir hypostasieren also unsere theoretischen Modelle um eben dieselben Modelle ontologisch zu begründen [...] das können wir aber nur tun, weil die sogenannten “Tatsachen” nichts anderes sind als Duplikat unseres theoretischen Modells — projiziert auf den metaphysischen Hintergrund einer absoluten Realität. Die ganze Prozedur läuft darauf hinaus, aufgrund von Wissenschaft mehr wissen zu wollen als Wissenschaft selbst...”[3]

Suppression of this “duplicate” would have the advantage of liberating us from some dualisms : the dualism of being and knowledge, of inside and outside, of subjective and objective. Let us pass Occam’s razor over all these useless entities and content ourselves with a pure phenomenism !

[3] E. KAESER : *Ein Dogma des kritischen Rationalismus*, Dialectica **28** (1974), p. 27.

4. *Reply to these critics*

Are we going to be convinced by this charge ? Some fact however should call for our attention.

Firstly : why is the non-philosopher inclined to realism ? Why does he believe that there is an externat world ? It is difficult to admit that this view is an unfortunate accident that has persisted — one wonders why. A more plausible hypothesis is that man has good *grounds* to be a realist. What are these grounds ?

Secondly : one distinguishes reality from illusion, dreams, appearance or error. This distinction is useful and efficient. Now we have just "proved" that they are impossible : since we do not know the real, we are not able to determine if a piece of knowledge corresponds to it or not, that is if this knowledge is true or false. How is it that we are able to detect that our senses sometimes deceive us ?

Realism is perhaps not a wholly obsolete position; it would allow some questions to be answered, provided it can be justified. For if the realist comes to a knowledge of the external world, he must explain how he achieves that.

5. *Search for a criterion of reality*

We will — perhaps provisionally — play the role of the idealist or of the positivist : we have no immediate knowledge of the "outside", we have only inner ideas, "schema". However, there must be something in these schema that allows us to say if such appearance corresponds to something real or not. For example : a stick appears broken : it is so, or is it only an illusion due to the refraction of light rays through water ? I see an object in some direction : is it really there or do I only see its image in a mirror ? In other words, there must be "criteria of reality", that allow us to accept or reject some idea inferred from experience as correspondmg to something real.

We thus remain in the domain of representation, of knowledge, but search for a criterion allowing us to recognise those ideas of which we can say that they are ideas "of something real".

More precisely it is not a criterion of reality that I shall propose, but rather a criterion of non-reality (in an analogous manner, Popper has replaced the positive criteria of confirmation by negative ones of falsification). There is no sufficient criterion that would allow us to declare something real.

(In the following I shall here try to coherently distinguish between :

- *experience*, and the adequacy of the schema to experience, which I will call suitability;
- on the other hand *reality*, or the real that applies to certain elements of the schema.

I shall not therefore be saying that the scholar encounters or is forced against reality, or that he measures the suitability of his theories by confronting them with reality.

Of course, reality supposes first suitability : only elements of a suitable schema can be said to be real.)

6. *Invariance*

The criterion of reality we are looking for could be *invariance*.[4] A concept or an attribute will be said to be invariant with respect to a certain change if this change does not affect the concept or attribute.

Let us take a model. The elements of a vector space are of course named 'vectors'.

These vectors can be referred to a coordinate system; components are then assigned to them : if the space has three dimensions, each vector is assigned a triplet of numbers representing its components in the chosen reference system.

The same set of vectors can be referred to various coordinate systems; when the system is changed, the components of a vector are changed too, although the vector itself remains unchanged.

The vector is therefore an invariant with respect to different reference systems. It is a quasi-metaphysical invariant, since a vector cannot be defined unless its components are given in some coordinate system.

But, if the vector space has been provided with a scalar product, invariant numbers can be assigned to such invariant vectors or to a pair of them. Thus, in any system, the scalar product of two vectors has the same value. That is to say that the length of a vector and the angle between two vectors do not depend on the chosen reference frame.

In this model, I would say that the vector is real. It has objective properties : its length and the angle it makes with other vectors. On the other hand the components, which depend on an arbitrary choice of reference frame, cannot be considered as real; they are relative to the chosen reference.

This example is only a model. I do no wish to say that vectors are more real than their components since vectors are mathematical entities to which the qualification "real" cannot be applied without caution. I only wish to suggest that there is the same relation between reality and appearances as there is between a vector and its components.

Another example, nearer daily life : when our eye is at a point P and we look in a given direction *n*, we see a perspective of a spatial object (that nearly matches the projection of the object from the point P on a plane perpendicular to *n*.)

If the point of view P is changed, the perspective changes, even though the spatial object (e.g. a cube) remains at rest — that is if its spatial coordinates don't change.

[4] E. Kaeser bas drawn my attention to a paper of M. Born in which he bas already stressed the importance of invariance as a criterion of reality (*Philosophical Quarterly* **3** (1953), 139).

From two perspectives the spatial object can be reconstructed : it suffices to state where the rays, issued from the same point, intersect. (This intersection exists for all points except those (A) such that A, P, P' lie on a straight line, P and P' being the two points of view. More than two perspectives may be needed in the case of an opaque object.) The spatial object allows the perspective to be reconstructed for all possible points and directions of view.

The spatial object is thus, so to speak, the invariant of all the perspectives. It acquires therefore the dignity of "reality", while the perspectives themselves are degraded to the rank of "appearances".

We can go even farther. Since the point of view means the position of the observer, of the subject, these perspectives can be said to be "subjective", while the coordinates of the spatial object (length of its edges, the angles between them) can be said to be 'objective', since they are independent of the subject's position, invariant with respect to the point of view and therefore proper to the object itself (one speaks then of *properties*).

What this example illustrates is, on one hand, the independence of the real from the subject, since different observers can be set at P and P'; but on the other hand the independence of the real from the various positions of the same observer, from the state of the subject, so to speak. This study will be developed in both directions.

7. *Two features of realism*

But, before this, it is perhaps good to stay a moment and to draw the lesson of this illustration.

The first remark, that can be surprising for the naive realist, is that *the real is not immediately given*; it is on the contrary *constructed*. No perspective shows the spatial object itself. Visually the spatial object is inferred from several perspectives, and it is inferred by postulating a schema, i.e. euclidean geometry. Experience is not the real; one can even say, at least primitively, experience is not real.

Second remark : realism is not only a theory of the object, but is equally a theory of the subject. The observer's point of view itself enters in the spatial schema, the eye of the subject has its spatial coordinates.

The realist, in every experience, tries to discriminate between the part belonging to the subject and that to the object; he distinguishes carefully what is due to a change in the subject from what can be ascribed to an alteration of the object. This distinction is not always easy, but is very important.

8. *Other invariances*

Let us now review other invariances.

a) We have briefly evoked invariance with respect to the *observer*. It is a regularly used criterion : if I doubt of the reality of what I have seen, I begin by asking other witnesses if they have seen the same. Nobody will raise difficulties about admitting that what is real can be stated by any fair observer.

It would be however a mistake to give an excessive importance to this criterion. Intersubjectivity is not objectivity : it has often happened that the majority was unanimously wrong and that one person alone was right against all (e.g. Heraclides Ponticus and Aristarchus of Samos, who claimed, in the IV and III centuries B.C., that the alternance of days and nights was due to the rotation of the earth about its own axis and even that the earth turned around the sun). Here again, the criterion is negative : it excludes the reality of certain phenomena, but does not prove it.

Incidentally, the importance of *measurement* to obtain an agreement between observers should be stressed. 'Big', 'long', 'hot' have no precise meaning; it depends on the reference chosen. In contrast, 'bigger than', 'longer than', 'hotter than', 'more luminous than' have meaning on which it is easy to agree, if one excepts some conflicts when the thing compared are too similar and become difficult to distinguish.

The introduction of a metric leads in the same direction : 'three times as long as' also has a precise meaning, acceptable for everyone. It is indeed the ratio between two lengths which is invariant, the number that measures the length itself being dependent on the chosen gauge. The gauge also must remain invariant, which often raises difficult problems. Already Plato stressed the importance of measurement to attain the real.[5]

b) We have also met the invariance with respect to the *state of the subject*, in particular with respect to his position. But there are other states : everyone knows the experiments in which the same tepid water seems hot or cold, depending on whether the hand had been previously dipped in cold or hot water, or the pink world after wearing green spectacles, or the spinning world after some pirouettes. In the same way, the visual world becomes double and shifted when you press on your eyeball : all these alterations, these variations of perception are assigned to a modification of the subject, not of the object.

c) Invariance with respect to the point of view can be considered together with invariance with respect to the *reference frame* (we have already met this in the vector-model). This invariance was the starting point and the guide of Einstein (principle of relativity : "the laws of physics must be such that they are valid in arbitrarily mowing reference frames".[6] This requirement is already clearly formulated in the first paragraph of the article in which he proposed the theory of special relativity :

Dass die Elektrodynamik Maxwells — wie dieselbe gegenwartig aufgefasst zu

[5] *Republic* **X**, 602*c*, *d*.

[6] *Ann. Phys.* **49** (1916).

werden pflegt — in ihrer Anwendung auf bewegte Körper zu *Asymetrien* führt, *welche den Phänomenen nicht anzuhaften scheinen*, ist bekannt. Man denke z.B. an die elektrodynamische Wechselwirkung zwischen einem Magneten und einem Leiter. Das beobachtbare Phänomen *hängt hier nur ab von der Relativbewegung* von Leiter und Magnet, während nach der üblichen Auffassung die beiden Fälle, dass der eine oder der andere dieser Körper der bewegte sei, streng voneinander zu trennen sind. Bewegt sich nämlich der Magnet und ruht der Leiter, so entsteht in der Umgebung des Magneten ein elektrisches Feld von gewissem Energiewerte, welches an den Orten, wo sich Teile des Leiters befinden, einen Strom erzeugt. Ruht aber der Magnet und bewegt sich der Leiter, so entsteht in der Umgebung des Magneten kein elektrisches Feld, dagegen im Leiter eine elektromotorische Kraft, welcher an sich keine Energie entspricht, die aber — Gleichheit der Relativbewegung voraussetzt — zu *elektrischen Strömen von derselben Grösse und demselben Verlauf* Veranlassung gibt, wie im ersten Falle die elektrischen Kräfte.[7]

(My italics. Einstein is reproaching electrodynamics for its non-invariance with respect to the reference frame, whereas the *facts* themselves are invariant. This paradox will be avoided by the definition of quadrivectors invariant after a spacetime transformation, the components of the quadrivectors depending on the chosen reference frame.)

d) Invariance with respect to the state of the subject should be considered together with invariance with respect to the *information* of the subject. It has been often stressed[8] that the probability of an event depends on the knowledge of the person evaluating the probability (or more precisely that part of the knowledge he decides to take into account). Different observers, diversely informed, can attribute different probabilities to the same event, and for the same observer the probability can suddenly change when he gets new information. Thus probability can by no means be considered as a real property of the event (unless one takes a standard knowledge or ignorance position, as one always does with urn or cards models).

Incidentally it was for this reason that quantum theory, being a probabilistic theory, was not considered by Einstein as a complete description. Contrary to what has often been alleged, Einstein's attitude was therefore perfectly consistent, even though he had been a revolutionary in relativity and a conservative with respect to quantum theory.

e) Invariance with respect to *time* sometimes raises delicate problems. There are indeed objects which keep rather well, at least during the relatively short periods in

[7] A. EINSTEIN : *Zur Elektrodynamik bewegter Körper*, *Ann. d. Phys.* **17** (1905).

[8] Marcel BOLL : *Les certitudes du hasard*, Coll. *Que sais-je ?*, PUF, Paris (1951), p. 25; F. GONSETH : *Philosophie des Sciences — Vue d'ensemble*, dans *La philosophie au milieu du vingtième siècle*, R. KLIBANSKY éd., La Nuova Italia Editrice, Firenze (1958); F. BONSACK : *Interprétations subjectivistes ou objectivistes ?*, *Lettres épistémologiques* **2** (Mai 1974).

which we observe them. And we often take them as paradigms of the real. But others evolve and change — objectively change, i.e. the object, which was invariant with respect to the point of view, or to different observers at time t_0, is also such at time t_1, the t_1-invariant being different from that of t_0. From this one concludes that the object has objectively changed. Even so, the distinction between an apparent change and a real change is not always easy; however, in most cases, an apparent change is reversible, whereas an irreversible change is generally real.[9]

When objects or situations are not invariant, one can search for magnitudes that are conserved, for example energy. Meyerson, in *Identité et Réalité*[10] has outlined this feature.

On the other hand, real changes — like the apparent ones — do not happen anyhow : there are rules and laws which govern them and, even if there is no conservation of the state, there is at least conservation of the law that links the successive states (differential equations). These laws are continuous (at least the macroscopic ones), so that two successive states are similar : if all is not conserved, there are at least traces left which allow one to recognize that the state at t_1 has resulted from the state at t_0. In respect of time therefore, continuity and variation according to constant laws would be more general than strict invariance.

In spite of all these limitations, the relative invariance of objects in time remains a valuable sign of their reality.

f) Invariance with respect to *motion*. Solid bodies, which have always been considered as specially real, approximately conserve their shape (i.e. the relative position of their points) during motion. This shows an invariance with respect to both time and space, an invariance that allows identification of a body through its successive states. This invariance brings no really new features to supplement pure spatial and temporal invariances.

g) Invariance with respect to *circumstances*.

A very demonstrative example is that of colour, which we would like to be an invariant property[11] of things. This it is not. Colour depends not only on the spectral composition of the light used, but even on the surrounding colours and their illumination. The same spot can appear yellow or brown according to the illumination of the surrounding frame. In current situations, physiological mechanisms adjust perception to the ambient lighting, so that the same object doesn't appear yellow by intense light and brown by semidarkness; it secures thus

[9] See POINCARÉ : *Des fondements de la géométrie*, Chiron, Paris (sans date; < 1928), p. 13-14.

[10] Vrin, Paris, 5th ed. (1951).

[11] The classical distinction between primary and secondary qualities would seem to rest on the higher level of invariance of the former (this is at least partially due to the fact that they are more easily measured), the secondary qualities seeming more dependant on particularities of the subject or the circumstances.

invariance of colour with respect to illumination. This is one instance among several others of an unconscious physiological mechanism elaborating sense data by extracting their invariants : the body itself takes the first steps on the way leading from raw data to the winning of the real.

(The same happens for stereoscopic vision : we do not perceive two images from which we consciously reconstruct a three-dimensional object; it is indeed the three-dimensional world itself that comes to our consciousness. Here again the nervous system has already made a major part of the way to the real.)

But let us come back to colour. We have already mentioned the effect of illumination. How can we obtain a notion of colour that is independent of ambient light ? We must direct our attention to the coefficient of reflexion for every wavelength, i.e. to the reflexion spectrum (if an opaque body is concemed). This invariant will allow one to reconstruct the perceived colour for every light whose spectral composition is known. But here again the characteristics of the subject must also be known i.e. the sensitivity of his receivers to the different wavelengths. If these data are available, one can even predict what a colour-blind person will be able to discriminate.

h) Invariance with respect to the *mode of perception.*

This is an invariance very often used as a criterion of reality.

It has been stressed that, among our senses, it is vision that most frequently allows illusion (because light signals must cross some distance before reaching our eyes, and on their path they can be deviated by refraction or reflexion.)

Control is made by another sense, for instance by touch : if no angle is felt, though one is seen when a stick is dipped in water, invariance of the stick is postulated and we conclude that sight has deceived us. This is no arbitrary decision in favour of touch; it rests not only on touch, but on the fact that the stick has none of the usual properties of a broken or even bent stick : the bend shifts if the stick is moved, it disappears when the stick is perpendicular to water and so on. The only way to restore the stick's invariance through all these experiences is to impugne the conclusions drawn from the visual data in some circumstances. This is readily accepted as physical laws established in other contexts explain why a straight stick must appear bent when dipped into water.

We establish a correspondence between messages obtained by the various senses in a given situation; thus the edge of a cube is seen as a straight line and as a projecting edge, but it can also be touched as a projecting edge without any bends, invariant over a certain length with respect to a straight translation, or as a boundary between two faces felt by touch as an uniform pressure invariant over some domain with respect to any translation in the plane. A one to one correspondence can thus be set up between the elements of the visual cube and those of the tactile one; this task is especially easy when the sensations are simultaneous i.e. when we see what we touch. Further : one can state that there is a correspondence between relations

linking these elements in both pictures : the number of edges that meet at each apex, the angle between the edges, the number of edges bordering one face, remain the same when we switch from visual to tactile perception. In the same way, if we may use a reference ruler and square, we can state both through touch and through vision that the angles or the edges are equal.

There is thus an isomorphism between the picture provided by sight and that provided by touch; a real object is in some sense the invariant of this isomorphism.

i) Invariance with respect to the *kind of experience* is closely related to that which we have just met; it lies however at a higher level of abstraction.

Thus the molecular and atomic hypotheses could appear hazardous as long as they only rested on the law of multiple proportions. However this is no longer the case since support has been found everywhere : in thermodynamics, in the kinetic theory of gases and in cristallography. It has been possible to evaluate by different methods and with concordant results the number of molecules in a grammolecule, to compute their velocity and their size, to discover their internal structure and their electrical properties. Their omnipresence under such diverse manifestations lead us to ascribe to them a reality status as an invariant of all these manifestations. And one could remain those who doubt of the atom's reality because one doesn't see them, that familiar objects are also constructed, abstracted from the appearances through which they show themselves. (There is a perfect continuity between what is seen — for instance brownian motion or optical diffraction gratings — and what is not seen — molecular motion and cristalline structure).

j) Invariance with respect to *language* (we rejoin here the theme of this colloquium).

It is a commonplace to acknowledge that the same state of fact can be expressed in various ways by using different languages and that even in a given language, it can be described with different words and sentences.[12]

One can say that the real is the invariant with respect to all linguistic modes used to describe it.

What is trivial for natural languages has less trivial consequences if one enlarges the meaning of the word "language" to *scientific theories* such as geometry.

The same domain of a plane — e.g. the inside of a circle — can be equally well described using euclidean geometry (with the usual definition of distances and angles) as by hyperbolic geometry (with less immediate definitions of distances and angles).

The invariants of both languages are not the same. They coincide for the relations of incidence and order, but congruent segments and angles in one language are not such in the other.

One could be tempted to conclude that reality depends on the language used, i.e. the contrary of the thesis defended here.

[12] I do not wish to discuss here the linguistic relativity defended by certain authors, for example by Benjamin Lee Whorf.

There are two possible ways of avoiding this difficulty.

The first, sometimes too restrictive, consists in ascribing to reality only that which is common to all schemata, to all languages (e.g. here only the relations of incidence and order). The second confers a privilege to one of the schemata (to one of the languages) claiming it more "natural" than the others.

Much could be said concerning this qualificative "natural".

It is sometimes possible to adjust several different schemata to the same domain by an appropriate choice of correspondences between the elements of the schema and those of experience (in our example of the inside of a circle by a change in the definitions of congruence).

But these correspondence rules must be justified : we will perhaps be prepared to accept that "distance" should be measured by a complex function of one or several distances (in the usual sense), in as far as these manipulations can be justified (for example corrections needed in consequence of physical influences such as temperature). However, manipulations which have to be carried out solely in order to accord a stubborn fact of experience to an ill-adapted theory will make the theory very suspect (e.g. the action of an ether-wind on the length of rods).

Often therefore when several different schema apply to the same domain, they do not all have the same value : those which require arbitrary correspondence rules are not considered satisfactory; preference will thus be given to the invariants of a theory that require no such artifices.

9. *Why this search for invariants ?*

One of the reasons is obvious : information must be stored such that room is not wasted, though care must be taken not to lose too much information. The information must be condensed such that it can be later retrieved, if not in its integrality, at least in that part in which we are interested.

However one knows, from information theory, how to compress information without loss : for this one must take advantage of constant or frequent bindings between elements, in other words discover the invariant or seldom varying element clusters.

The example of invoking a spatial object well illustrates its advantages : instead of storing all perspectives, one stores only the coordinates of the reconstructed object. Not only all the perspectives under which the object has appeared, but all the future ones, under which it could and possibly will appear, can be derived from this limited amount of information. This shows a second reason for the search for invariants. They not only allow one to summarize past experience, but also to foresee new experiences, predicting how a given invariant will appear in given circumstances.

There is also the interest of dividing a global situation into different objects, and even to split a perceptive situation into a subject and an object. An example will provide a better understanding. Take a subject S and an object O. Let us suppose

that the way O appears to S depends on the different states of O : O_1, O_2, O_3 ... O_m and on the different states of S : S_1, S_2, S_3 ... S_n.

If I do not dissociate the perceptive situation into subject and object, I must store $m \cdot n$ such situations. On the other hand, if I dissociate subject and object, and if I am able to restore, from the states of the subject and object, what S_i perceives when the object is in the state O_j, it is sufficient, for the same amount of information, to memorize only $m + n$ states. Moreover it is not necessary to have experienced all the combinations : it is possible to reconstruct the missing ones.

Thus if we split a global situation into several elements which by combination will allow a reconstruction of the situation, we replace somehow products by sums. This becomes increasingly advantageous with increasing numbers of elements and of states that these elements can adopt. (Meccano principle).

10. *The invariant limit*[13]

Taking up this aspect of invariance, I will start again from a mathematical model.

Everyone knows that there is no rational number that gives two when squared. Thus $\sqrt{2}$ doesn't exist in the domain of rational numbers.

But we can construct a sequence of rational numbers (e.g. 1, 1.4, 1.41, 1.414, 1.4142, 1.41421...) that approach $\sqrt{2}$ as closely as desired. Such a sequence is said to converge towards $\sqrt{2}$, or $\sqrt{2}$ is the limit of this sequence.

The limit L of a sequence can be defined as follows : L is the limit of a sequence of numbers if all the terms, following the n^{th} term (rank "n") are contained within an interval centered on L. Decreasing the interval obliges us to look for a n farther in the sequence — n is increased — but if the sequence converges, such a n can always be found.

Suppose we take 1.414 as candidate for the limit. Everything goes alright up to the term 1.414 : one comes nearer to the "limit", but afterwards, one diverges from it. 1.414 can thus play the role of a limit *up to some rank*; after that, one must change the "limit" that will operate little further, and so on.

One can therefore manage with rational numbers, provided one defines open, revisable limits that will work upto some rank, but will fail further on.

But this solution has never been in favour among mathematicians : for them, a limit must be invariant. They postulate therefore that there exists, beyond all the approximations of $\sqrt{2}$ by rational numbers, an irrational number exactly equivalent to $\sqrt{2}$, a number that no one has ever been able to write in any notation since all notations allow only rational numbers (unless one admits infinite sequences of decimals that nobody can ever write).

[13] The meaning of the word 'limit' is here enlarged, what the mathematicians usually calI 'limit' being named 'invariant limit'.

Let us go a bit farther with the mathematical model.

The definition we have given of a 'limit' has a serious flaw when the limit is irrational : the limit is unknown and therefore it is impossible to compute the gap between a term of the sequence and the limit.

For this reason, Cauchy has given another criterion of convergence for a sequence that no longer makes use of the notion of limit. He calls a sequence 'convergent' if, for as small an "ε" as one cares to choose, one can indicate a rank N such that beyond this rank the difference between any two terms of the sequence is smaller than ε. This criterion has the advantage of using only terms of the sequence (rational terms, if the sequence is rational) and not an often inaccessible and inexpressible limit. The criterion could be formulated in a slightly different way : A sequence converges if, for an arbitrarily small ε, a rational "open limit" can be found such that from some rank, the difference between any term and this "limit" is smaller than ε. Of course, if a smaller ε is chosen this "open limit" will have to be changed, but it will again be possible to find a rational one.

I postulate that the real is the invariant limit of our schemata, of our theories. We do not know this limit, but each state of knowledge can be considered as the open limit of preceding states; the sequence therefore provisionally converges.

When we restrict ourselves to a given time it is not immediately obvions why the schema should be duplicated, hypostatized into a metaphysical reality that would be behind them.

Yet this is no longer the case if we take an historical perspective — be it collective or individual history. Our knowledge evolves, some schemata are replaced by others that are more efficient, more precise and generally more suitable.

If the real is characterized by invariance, one cannot be satisfied with a growing real, with a real that changes with time. The realist — and he is therein fully consistent with his fundamental options — postulates the real's invariance with respect to all subjects during of history, and to the changes of the subject himself, provided of course that the real has not objectively changed.

I am quite aware that this "convergence" of theories towards a real has been disputed, especially by P.K. Feyerabend. So I have to make clear my stand-point towards this criticism.

The mathematician knows very well that in order to speak of convergence, he must first define a topology, i.e. something that allows him to state if two elements are neighbouring. An easy way to define neighbourhoods is to provide oneself with a metric, to define a distance (open balls as neighbourhoods).

How is the distance between two theories measurable ? One could compare their logical structures, and say that two theories are neighbouring when they have a similar logical structure. This is what Feyerabend does though it is still necessary to clarify what one means by "neighbouring logical structure" and define some "logical distance".

One could also define a "psychological distance", in order to measure the magnitude of the gap to be crossed to pass for instance from classical mechanics to relativity. This is the direction which Kuhn takes.

These two topologies are legitimate : it is certainly interesting to show how far a theory can be, in its fundamental concepts, from the theory it replaces. But they are very disappointing if one wishes to define a convergence : the distance between successive theories is so large that they have been considered 'incommensurable'.

This is not at all the position that mathematics has taken. One has not said "two functions are neighbouring when their 'logical structures' are analogous". On the contrary, one does not hesistate to write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \ldots$$

The structures of the two members are obviously incommensurable : on one side we have a transcendental trigonometric function, on the other a series of powers.

Yet these functions are not only neighbouring, but equal because, for an arbitrarily large, but finite range of x's, and as small the maximum allowed difference ε as desired, it is always possible to take a sufficient number of terms to make the difference smaller than ε over the whole range.

The theorem of approach also shows the extent to which logical structures as incompatible as euclidean, hyperbolic and elliptical geometry can be neighbouring. ("On any domain of the projective plane, bounded for an euclidean metric, one can define hyperbolic and elliptic metrics which differ from the euclidean one by as little as wanted over the whole domain".[14])

On the other hand, the physicist is constantly using approximations that are only valid in certain circumstances (for example if velocities are small compared to the velocity of light). These approximations are also incompatible with the exact law (in the sense of Feyerabend). They are neverless neighbouring in their results (at least over a certain range).

One can thus very well speak of convergence of the schema or the theories toward the real, *provided one defines a suitable topology* — a topology of *values* or outcomes. Two theories are neighbouring over some range if they provide similar results and predictions, however large the "distance" between their logical structures or the psychological gap to be overcome in passing from one to the other. Using a metaphor, one could say that the distance should not be measured between roots, but between the twigs.

You will perhaps ask me if this reality, these invariants which I postulate, are

[14] F. GONSETH : *La Géométrie et le problème de l'espace*, Le Griffon, Neuchâtel (1955), T. **VI**, p. 57.

exterior to the subject or if they have to be considered in the idealistic way as thought invariants, as pictures of the real.

Frankly, this question doesn't seem to me to have the importance that has been often attributed to it. If I had not some reticence to using the word 'metaphysical' in a depreciatory sense, I would say that it is metaphysical. The "realism" defended here claims to be metaphysically neutral : even from an idealistic point of view one can consider the search invariants to be useful. Making place, among the representations of things, for a special thing that is the subject (not the subject I, theater of all representations, but a subject II, thought by subject I[15]), one might usefully construct a schema of sensation as an interaction between objects and the subject II, and distinguish, in these schematic sensations, what belongs to the subject II and what to the object. On the other hand, we have said that the real is constructed by thought and this can only please the idealist.

Remarks and replies to criticisms of my paper

1. I am aware of having restricted the meaning of the word 'reality', with the advantages and the drawbacks brought about by such a restriction. It is convenient to have a term with a well-defined meaning, allowing a secure usage and discussions without misunderstanding. But some will perhaps regret the meanings I have had to give up. I deal here only with that aspect of reality opposed to knowledge, appearance or illusion. A different aspect could have been chosen : that reality against which we fight, that restricts our arbitrariness, that answers with a yes or no the questions we put to it, i.e. what I have called 'experience'.

The choice being made, one has to remain consistent and to accept its consequences, paradoxical though they may be : my reality is an element of the schema, it is constructed; and experience is not "real" — at least not immediately.

One can however wonder if it could not be possible to find a common denominator for these two meanings of the word 'reality'.

2. This paper calls for another to develop paragraph 11 and the distinction outlined by Gilson and Blanché between the level of knowledge and that of reality. The present paper is thus deprived of a complementary view that would make some claims more acceptable. But one cannot deal with everything at the same time.

Other parts should be developed too, especially that concerning invariance with respect to language.

3. I have perhaps misused, as I often do, models, pictures, parables and analogies, striving to be precise in the chosen example but often neglegting the adaptation to other examples and the problems it can raise. It is a deliberate choice : personally

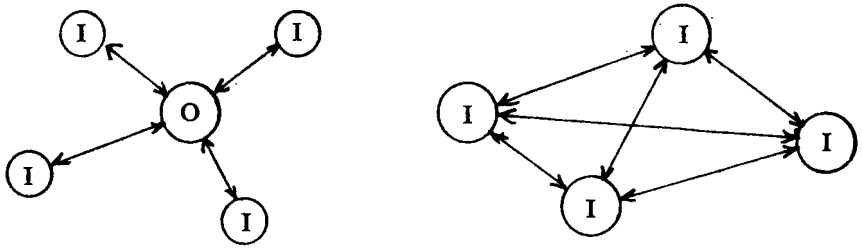
[15] See Robert BLANCHÉ : *Les attitudes idéalistes*, PUF, Paris (1949).

I prefer a precise idea which I am sure really applies in some domain, however restricted this might be, rather than general ideas which are often so abstract that they possibly cannot be applied anywhere efficiently.

4. One could object that I cannot manage to free myself habits of thought, certainly current, but nevertheless unjustified. Why should we postulate the existence of a limit ? Would it not be sufficient to define an equivalence relation between sequences that have the same limit ? Or could we not be satisfied with open limits ? When we have different models of a structure, must we hypostatize it into an abstract structure or should we privilege a particular model and elicit the rules allowing us to reduce to it ?

To take up the vector model again : is it really sound to postulate an invariant being the vector ? Could we not be content with the conversion rules for passing from one coordinate system to another, saying — as is sometimes done — a vector is that which is converted by following these rules ? Is it not this fact, that it changes according to these rules — and not arbitrarily — that guarantees its reality ?

Schematically : should we prefer a structure of type A or of type B ?



A B

where O stands for the original, the invariant, a spatial object, an abstract entity and I for an image, copy, appearance or various forms.

Frankly, I am not sure that it matters so much whether one choses one or the other option. They are perhaps two ways of saying the same thing. To set one against the other would be to return to a quarrel over universals, which I fear unlikely to be fruitful.

I thank Mr. P. Draper for his careful revision of my English text.